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INSTABILITY OF THE QUIESCENT STATE OF AN IDEAL CONDUCTING MEDIUM IN A MAGNETIC FIELD

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The linear stability of the quiescent states of an ideal compressible medium with infinite conductivity in a magnetic field is studied. It is shown by Lyapunov's direct method that these quiescent states are unstable relative to small spatial perturbations, which decrease the potential energy (the sum of the internal energy of the medium and the energy of the magnetic field in this case). Two-sided exponential estimates of perturbation growth are obtained; the exponents in these estimates are calculated using the parameters of the quiescent states and the initial data for perturbations. A class of the most rapidly growing perturbations is separated and an exact formula to determine the rate of their increase is derived. An example is constructed of the quiescent states and the initial perturbations whose linear stage of evolution in time occurs in correspondence with the estimates. From the mathematical viewpoint, our results are preliminary, because the existence theorems for the solutions of the problems considered are not proved.

1. Formulation of the Exact Problem. The spatial motions of an ideal compressible conducting medium in the magnetic field are considered [1]. It is assumed that these motions occur in the domain τ with a fixed, ideally conducting boundary $\partial\tau$ and are described by the following equations [2]:

$$\rho Dv_i = -\frac{\partial p}{\partial x_i} + \frac{h_k}{4\pi}\Big(\frac{\partial h_i}{\partial x_k} - \frac{\partial h_k}{\partial x_i}\Big), \quad D\rho + \rho\frac{\partial v_i}{\partial x_i} = 0, \quad Dh_i = h_k\frac{\partial v_i}{\partial x_k} - h_i\frac{\partial v_k}{\partial x_k},$$
$$\frac{\partial h_i}{\partial x_i} = 0, \quad Ds = 0, \quad D = \frac{\partial}{\partial t} + v_i\frac{\partial}{\partial x_i}, \quad e = e(\rho, s), \quad de = T\,ds - pd\Big(\frac{1}{\rho}\Big). \tag{1.1}$$

Here ρ, $\boldsymbol{v} = (v_1, v_2, v_3)$, p, s, e, and T are the density, velocity, pressure, entropy, internal-energy, and temperature fields, respectively, $\boldsymbol{h} = (h_1, h_2, h_3)$ is the magnetic field, $\boldsymbol{x} = (x_1, x_2, x_3)$ are the Cartesian coordinates, and t is the time. With other conditions not specified, summation is performed from 1 to 3 over the repeat indices.

It is assumed that the no-slip conditions

$$v_i n_i = 0, \qquad h_i n_i = 0 \tag{1.2}$$

are satisfied at the boundary $\partial\tau$. Here $\boldsymbol{n} = (n_1, n_2, n_3)$ is the unit external normal to $\partial\tau$. The second condition in (1.2) means that the magnetic field is concentrated entirely inside the region τ and does not leave its boundaries.

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The initial data for the boundary-value problem (1.1), (1.2) are set in the form

$$\boldsymbol{v}(\boldsymbol{x},0) = \boldsymbol{v}^0(\boldsymbol{x}), \qquad \rho(\boldsymbol{x},0) = \rho^0(\boldsymbol{x}), \qquad \boldsymbol{h}(\boldsymbol{x},0) = \boldsymbol{h}^0(\boldsymbol{x}), \qquad s(\boldsymbol{x},0) = s^0(\boldsymbol{x}). \tag{1.3}$$

Here the functions $\rho^0(\boldsymbol{x})$ and $s^0(\boldsymbol{x})$ are arbitrary, and, together with the function $\boldsymbol{h}^0(\boldsymbol{x})$, the function $\boldsymbol{v}^0(\boldsymbol{x})$ should reduce the fourth equation of system (1.1) to an identity, on the one hand, and guarantee the satisfaction of the boundary conditions (1.2), on the other hand. We assume that all the fields have a sufficient degree of smoothness.

It is worth noting that the mixed problem (1.1)–(1.3) is the mathematical model of a plasma installation whose magnetic system ensures ideal confinement conditions for a plasma which is in contact with the casing surface. A study of this problem is necessary to examine a more general and realistic (from the physical viewpoint), problem, namely, the problem simulating a plasma installation in which the ideal confinement conditions for a plasma separated from the casing by a vacuum interlayer (in accordance with the thermoinsulation requirement) are realized [3].

Direct calculations show that the energy integral

$$E_1 = K_1 + \Pi_1 = \text{const}, \qquad 2K_1 = \int\limits_\tau [\rho v_i v_i]\, d\tau,$$
$$\Pi_1 = \int\limits_\tau \Big[\rho e(\rho, s) + \frac{1}{8\pi} h_i h_i\Big]\, d\tau, \qquad d\tau = dx_1\, dx_2\, dx_3. \tag{1.4}$$

is preserved on the solutions of the initial boundary-value problem (1.1)–(1.3).

The exact steady-state solutions of problem (1.1)–(1.3)

$$\boldsymbol{v} = 0, \quad \rho = \rho_0(\boldsymbol{x}), \quad p = p_0(\boldsymbol{x}), \quad \boldsymbol{h} = \boldsymbol{h}_0(\boldsymbol{x}), \quad s = s_0(\boldsymbol{x}), \tag{1.5}$$

which correspond to the quiescent states of an ideal compressible medium with infinite conductivity in the magnetic field, satisfy the equations

$$\frac{1}{4\pi} h_{0k}\Big(\frac{\partial h_{0i}}{\partial x_k} - \frac{\partial h_{0k}}{\partial x_i}\Big) = \frac{\partial p_0}{\partial x_i}, \quad \frac{\partial h_{0k}}{\partial x_k} = 0, \quad e = e_0(\rho_0, s_0), \quad p_0 = \rho_0^2 \frac{\partial e}{\partial \rho}(\rho_0, s_0) \ \text{in}\ \tau \tag{1.6}$$

and the condition

$$h_{0i} n_i = 0 \quad \text{on} \quad \partial\tau. \tag{1.7}$$

The goal of the further consideration is to show the instability of the quiescent states (1.5)–(1.7) relative to small spatial perturbations.

2. Formulation of the Linearized Problem. To reach our goal, the initial boundary-value problem (1.1)–(1.3) is linearized on the exact steady-state solutions (1.5)–(1.7). As a result, we obtain the system of equations of motion

$$\rho_0 \frac{\partial v_i'}{\partial t} = -\frac{\partial p'}{\partial x_i} + \frac{h_{0k}}{4\pi}\Big(\frac{\partial h_i'}{\partial x_k} - \frac{\partial h_k'}{\partial x_i}\Big) + \frac{h_k'}{4\pi}\Big(\frac{\partial h_{0i}}{\partial x_k} - \frac{\partial h_{0k}}{\partial x_i}\Big), \quad \frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x_k}(\rho_0 v_k') = 0,$$
$$\frac{\partial h_i'}{\partial t} = \frac{\partial}{\partial x_k}(v_i' h_{0k} - v_k' h_{0i}), \quad \frac{\partial h_k'}{\partial x_k} = 0, \quad \frac{\partial s'}{\partial t} + v_k' \frac{\partial s_0}{\partial x_k} = 0, \tag{2.1}$$
$$p' = c_0^2 \rho' + \rho_0^2 s' \frac{\partial^2 e}{\partial \rho\, \partial s}(\rho_0, s_0), \qquad c_0^2 = \rho_0\Big(2\frac{\partial e}{\partial \rho}(\rho_0, s_0) + \rho_0 \frac{\partial^2 e}{\partial \rho^2}(\rho_0, s_0)\Big),$$

which determines the evolution of the small perturbations of the velocity $\boldsymbol{v}'$, density ρ', pressure p', entropy s', and magnetic field $\boldsymbol{h}'$ in the region τ with time. This system is supplemented by the no-slip conditions

$$v_i' n_i = 0, \qquad h_i' n_i = 0, \tag{2.2}$$

which are posed at the boundary $\partial\tau$, and by the initial data

$$\boldsymbol{v}'(\boldsymbol{x},0) = \boldsymbol{v}'^0(\boldsymbol{x}), \quad \rho'(\boldsymbol{x},0) = \rho'^0(\boldsymbol{x}), \quad \boldsymbol{h}'(\boldsymbol{x},0) = \boldsymbol{h}'^0(\boldsymbol{x}), \quad s'(\boldsymbol{x},0) = s'^0(\boldsymbol{x}); \tag{2.3}$$

the restrictions similar to those adopted earlier for the functions $\boldsymbol{v}^0(\boldsymbol{x})$ and $\boldsymbol{h}^0(\boldsymbol{x})$ from (1.3) are imposed on the functions $\boldsymbol{v}'^0(\boldsymbol{x})$ and $\boldsymbol{h}'^0(\boldsymbol{x})$. Hereafter, the primes at the quantities indicating the perturbation fields, which distinguish them from the full solutions of problem (1.1)–(1.3), are omitted.

The instability of any of the quiescent states (1.5)–(1.7) to small spatial perturbations may be considered proved only in the case where at least one perturbation that exponentially increases in time is found. In view of this, it is advisable to narrow the domain of search for this perturbation. Our further consideration is concerned with a class of motions in which the perturbations of the entropy of liquid particles (the Lagrangian perturbations of the entropy field) are equal to zero. In other words, it is assumed that the entropy of each liquid particle does not vary during perturbations, and the perturbations are assumed to be the displacements of the particles from their equilibrium positions. The simplest way to describe these perturbations is to introduce the field of Lagrangian displacements $\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{x}, t) = (\xi_1, \xi_2, \xi_3)$ [4] for which the relation

$$\frac{\partial \xi_i}{\partial t} = v_i \tag{2.4}$$

is satisfied.

Using the field $\boldsymbol{\xi}$, we write the linearized initial boundary-value problem (2.1)–(2.3) as follows:

$$\begin{gathered}
\rho_0 \frac{\partial^2 \xi_i}{\partial t^2} = -\frac{\partial p}{\partial x_i} + \frac{h_{0k}}{4\pi}\Big(\frac{\partial h_i}{\partial x_k} - \frac{\partial h_k}{\partial x_i}\Big) + \frac{h_k}{4\pi}\Big(\frac{\partial h_{0i}}{\partial x_k} - \frac{\partial h_{0k}}{\partial x_i}\Big), \\
\rho = -\frac{\partial}{\partial x_k}(\rho_0 \xi_k), \qquad h_i = \frac{\partial}{\partial x_k}(\xi_i h_{0k} - \xi_k h_{0i}), \qquad s = -\xi_k \frac{\partial s_0}{\partial x_k}, \\
p = c_0^2 \rho + \rho_0^2 s \frac{\partial^2 e}{\partial \rho\, \partial s}(\rho_0, s_0), \qquad c_0^2 = \rho_0\Big(2\frac{\partial e}{\partial \rho}(\rho_0, s_0) + \rho_0 \frac{\partial^2 e}{\partial \rho^2}(\rho_0, s_0)\Big) \ \text{ in } \ \tau, \\
\xi_i n_i = 0, \qquad h_i n_i = 0 \ \text{ on } \ \partial\tau, \qquad \boldsymbol{\xi}(\boldsymbol{x}, 0) = \boldsymbol{\xi}^0(\boldsymbol{x}), \quad \boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}^0(\boldsymbol{x}).
\end{gathered} \tag{2.5}$$

For the solutions of problem (2.4), (2.5), the linear analog of the energy integral holds:

$$\begin{gathered}
E = K + \Pi = \text{const}, \qquad 2K = \int_\tau [\rho_0 v_i v_i]\, d\tau, \\
2\Pi = \int_\tau \Big\{ -p \frac{\partial \xi_k}{\partial x_k} + \frac{h_i}{4\pi}\Big[h_i - \xi_k\Big(\frac{\partial h_{0k}}{\partial x_i} - \frac{\partial h_{0i}}{\partial x_k}\Big)\Big]\Big\}\, d\tau.
\end{gathered} \tag{2.6}$$

One can show that the second variation of the functional Π_1 (1.4), which is written in appropriate notation, coincides with the functional Π, and its first variation vanishes on the quiescent states (1.5) by virtue of conditions (1.6) and (1.7).

If, for all admissible fields of Lagrangian displacements $\boldsymbol{\xi}$ (2.4), the inequality $\Pi \geqslant 0$, which corresponds to reaching a minimum of the functional Π_1 on the exact steady-state solutions (1.5)–(1.7) of problem (1.1)–(1.3), is valid, the stability of the quiescent states (1.5)–(1.7) relative to small spatial perturbations follows from the nondependence of the functional E on time. In essence, this result is one of the forms of the Lagrange theorem [5–7] on the equilibrium stability of the mechanical system in the presence of a minimum of the potential energy in it.

The Lagrange theorem will be inverted below, i.e., the instability of the quiescent states (1.5)–(1.7) to small spatial perturbations will be shown provided that the functional Π_1 does not reach its minimum value on them. In terms of the field of Lagrangian displacements $\boldsymbol{\xi}$, this means that there is an initial field $\boldsymbol{\xi}^{0*}(\boldsymbol{x})$ (2.5) which has the following important property:

$$\Pi(0) < 0 \quad \text{if} \quad \boldsymbol{\xi} = \boldsymbol{\xi}^{0*}(\boldsymbol{x}). \tag{2.7}$$

For other initial fields of Lagrangian displacements $\boldsymbol{\xi}^0(\boldsymbol{x})$, the inequality (2.7) can be replaced by an opposite inequality, i.e., the quiescent states (1.5)–(1.7) are the infinite-dimensional analog of the "saddle" point of the functional Π_1 (1.4).

3. Lyapunov Functional. According to [8–10], we introduce the auxiliary functional

$$M = \int_\tau [\rho_0 \xi_i \xi_i]\, d\tau, \tag{3.1}$$

the double differentiation of which in time and subsequent transformations with the use of (2.4)–(2.6) yields the relation

$$\frac{d^2M}{dt^2} = 4(K - \Pi) = 8K - 4E,$$

called a virial equality [4]. In turn, multiplying this equality by an arbitrary constant factor λ and subtracting from (2.6), we obtain the equation

$$\begin{gathered} \frac{dE_\lambda}{dt} = 2\lambda E_\lambda - 4\lambda K_\lambda, \qquad E_\lambda = K_\lambda + \Pi_\lambda, \qquad 2\Pi_\lambda = 2\Pi + \lambda^2 M, \\ 2K_\lambda = 2K - \lambda \frac{dM}{dt} + \lambda^2 M = \int_\tau \rho_0 \Big(\frac{\partial \boldsymbol{\xi}}{\partial t} - \lambda \boldsymbol{\xi}\Big)^2 d\tau. \end{gathered} \tag{3.2}$$

If one sets $\lambda > 0$, by virtue of the nonnegativity of the functional K_λ, the differential inequality

$$\frac{dE_\lambda}{dt} \leqslant 2\lambda E_\lambda$$

follows from (3.2), the integration of which allows one to obtain the relation

$$E_\lambda(t) \leqslant E_\lambda(0) \exp{(2\lambda t)} \tag{3.3}$$

which is of primary importance for our consideration.

It is noteworthy that the inequality (3.3) is valid for any solutions of problem (2.4), (2.5) and for any positive values of the parameter λ. In addition, the derivation of this inequality does not require any restrictions to be imposed on the sign of the functional Π(2.6).

The inequality (3.3) indicates that the functional E_λ varies monotonically with time. This circumstance enables us to regard it as the Lyapunov functional [5, 8].

4. Upper and Lower Estimates. Setting the initial data for the fields of Lagrangian displacements $\boldsymbol{\xi}$ and the perturbations of the velocity field $\boldsymbol{v}$ (2.4), (2.5) appropriately, with the use of the inequalities (3.3) one can obtain two-sided exponential estimates of the increase in small spatial perturbations of the quiescent states (1.5)–(1.7) and to derive an exact formula for calculation of the rate of increase of the most rapidly growing perturbations.

Indeed, let the initial field of Lagrangian displacements $\boldsymbol{\xi}^0$ be such that condition (2.7) is satisfied for this field. Taking into account that the fields of Lagrangian displacements $\boldsymbol{\xi}$ and the perturbations of the velocity field $\boldsymbol{v}$ are set at the initial moment irrespective of each other, as the latter, we can choose the functions $\boldsymbol{v}^0$ which satisfy the inequality $K(0) < |\Pi(0)|$. In this case, according to (3.2), the functional $E_\lambda(0)$ will be nothing but the second-order polynomial relative to λ with a positive factor $M(0)$ (3.1) for λ^2 and a negative constant term $E(0)$ (2.6):

$$E_\lambda(0) = E(0) - \frac{\lambda}{2}\frac{dM}{dt}(0) + \lambda^2 M(0). \tag{4.1}$$

Let $\lambda > 0$; then it follows from (4.1) that, on the interval

$$0 < \lambda < \Lambda_1 = B + C^{1/2} \qquad \Big(B = \frac{1}{4M(0)}\frac{dM}{dt}(0), \quad C = B^2 - \frac{E(0)}{M(0)}\Big), \tag{4.2}$$

the relation

$$E_\lambda(0) < 0 \tag{4.3}$$

holds.

The inequalities (3.3) and (4.3) show that the solutions of the initial boundary-value problem (2.4), (2.5) increase exponentially in time.

If $\lambda = \Lambda_1 - \delta$ with any δ from the interval $]0, \Lambda_1[$, relation (3.3) takes the form

$$E_{\Lambda_1-\delta}(t) \leqslant E_{\Lambda_1-\delta}(0) \exp[2(\Lambda_1 - \delta)t] \qquad [E_{\Lambda_1-\delta}(0) < 0]. \tag{4.4}$$

Using the definition of the functionals E_λ, K_λ, and Π_λ (3.2), one can derive the inequality $E_\lambda(t) > \Pi(t)$, which allows one to put relation (4.4) into the form

$$\Pi(t) < E_{\Lambda_1-\delta}(0) \exp[2(\Lambda_1 - \delta)t]. \tag{4.5}$$

Using the additional functional

$$J(t) = \int_\tau \left\{\rho^2 + s^2 + \left(\frac{\partial \xi_i}{\partial x_i}\right)^2 + h_i h_i + \xi_i \xi_i\right\} d\tau, \tag{4.6}$$

we transform the inequality (4.5) to the more informative relation

$$J(t) > |cE_{\Lambda_1-\delta}(0)| \exp[2(\Lambda_1 - \delta)t] \tag{4.7}$$

(c is the known constant), from which one can conclude that the parameter $\Lambda_1 - \delta$ of (4.2) and (4.4) gives the lower estimate of the increments of the solutions of problem (2.4), (2.5).

The estimate (4.7) is essentially improved if the initial perturbations of the velocity field $\boldsymbol{v}^0$ (2.5) are related to the initial field of Lagrangian displacements $\boldsymbol{\xi}^{0*}$ (2.5) and (2.7) as follows:

$$\boldsymbol{v}^0(\boldsymbol{x}) = \lambda \boldsymbol{\xi}^{0*}(\boldsymbol{x}). \tag{4.8}$$

Indeed, in the presence of the coupling (4.8) it follows from of (3.2) that

$$K_\lambda(0) = 0, \qquad E_\lambda(0) = \Pi_\lambda(0). \tag{4.9}$$

Assuming $\lambda > 0$, according to (4.1) and (4.9), it is easy to see that the relation $\Pi_\lambda(0) < 0$ holds on the interval

$$0 < \lambda < \Lambda = \left(-\frac{2\Pi(0)}{M(0)}\right)^{1/2}. \tag{4.10}$$

If $\lambda = \Lambda - \delta_1$ with any δ_1 from the interval $]0, \Lambda[$, with allowance for (4.9) the inequality (3.3) can be reduced to the relation

$$E_{\Lambda-\delta_1}(t) \leqslant \Pi_{\Lambda-\delta_1}(0) \exp[2(\Lambda - \delta_1)t] \qquad (\Pi_{\Lambda-\delta_1}(0) < 0). \tag{4.11}$$

From (4.6) and (4.11), we obtain the inequality

$$J(t) > |c_1 \Pi_{\Lambda-\delta_1}(0)| \exp[2(\Lambda - \delta_1)t] \tag{4.12}$$

(c_1 is the known constant), which shows that the parameter $\Lambda - \delta_1$ of (4.10) and (4.11) gives the lower estimate of the increments of the solutions of the initial boundary-value problem (2.4), (2.5), (4.8).

A comparison of the estimates (4.7) and (4.12) shows that the solutions of problem (2.4), (2.5), the initial data for which satisfy condition (4.8), increase more rapidly than other perturbations.

Moreover, one can show that the perturbations with initial data (4.8) are the most dangerous, because the most rapid growth of the solutions of problem (2.4), (2.5) is observed for

$$\Lambda^+ = \sup_{\boldsymbol{\xi}^{0*}(\boldsymbol{x})} \Lambda. \tag{4.13}$$

To do this, it is necessary to obtain an estimate that restricts from above the increase in the small spatial perturbations of the quiescent states (1.5)–(1.7). For this, as a parameter λ, we use a positive number which is greater than Λ^+ in magnitude. Then, for all the possible initial fields of Lagrangian displacements $\boldsymbol{\xi}^0$ (2.5), the relation $\Pi_\lambda(0) > 0$ is valid. Hence, the functional E_λ (3.2) is also positive definite for all the possible initial fields of Lagrangian displacements $\boldsymbol{\xi}^0$ and the perturbations of the velocity field $\boldsymbol{v}^0$ (2.4) and (2.5).

This means that, for $\lambda = \Lambda^+ + \varepsilon$ ($\varepsilon > 0$), the relation

$$E_{\Lambda^+ + \varepsilon}(t) \leqslant E_{\Lambda^+ + \varepsilon}(0) \exp\left[2(\Lambda^+ + \varepsilon)t\right],$$

follows from the basic inequality (3.3). Using the inequalities $\Pi_{\Lambda^+}(t) \geqslant 0$, the latter relation can be rewritten in the more obvious form

$$2K_{\Lambda^+ + \varepsilon}(t) + \varepsilon(2\Lambda^+ + \varepsilon)M(t) \leqslant 2E_{\Lambda^+ + \varepsilon}(0) \exp\left[2(\Lambda^+ + \varepsilon)t\right]. \tag{4.14}$$

It follows from relation (4.14) that the parameter $\Lambda^+ + \varepsilon$ of (4.10) and (4.13) gives the upper estimate of the increments of the solutions of the initial boundary-value problem (2.4), (2.5).

A comparison of the inequalities (4.12) and (4.14) allows one to conclude that the parameter Λ^+ estimates the rate of increase of the solutions of problem (2.4), (2.5), (4.8) from below and from above:

$$\Lambda^+ - \delta_1 \leqslant \omega_* \leqslant \Lambda^+ + \varepsilon. \tag{4.15}$$

The estimate (4.15) shows that the solutions of the initial boundary-value problem (2.4), (2.5), whose increment is equal to Λ^+ (4.13), increase most rapidly.

Hence, if condition (2.7) is satisfied, having calculated the value of Λ^+ by formulas (4.10) and (4.13), we can answer the following question: for what characteristic time will the quiescent states (1.5)–(1.7) of an ideal compressible medium with infinite conductivity in the magnetic field "deteriorate"?

We note that the linear stability of the magnetohydrodynamic equilibrium of an ideally conducting plasma with the equation of state in the form of a Poisson adiabat was investigated by the authors in [11]. The basic drawback of the sufficient conditions for plasma confinement in the magnetic trap obtained in [1] lies in that the construction of the corresponding Lyapunov functional includes the first integrals of helicity, As a matter of fact, they led to an implicit elimination of the zero-helicity perturbations from consideration, which, nevertheless, decrease the effective potential energy. The results of the present study are free from this disadvantage.

Another important circumstance, to which attention should be paid, is that the above-described technique of obtaining two-sided exponential estimates of the increase in small spatial perturbations can be applied also to the problem of the instability of spatially-periodic magnetic fields located in a quiescent boundless ideal compressible medium of infinite conductivity. Here the domain τ is the analog of elementary cells of these magnetic fields, and its boundary $\partial\tau$ is the analog of the cell surface.

5. Example. Let the inviscid compressible, ideally conducting medium filling the boundless space be at rest in a magnetic field of the form [12–16]

$$\boldsymbol{h}_0 = A(\cos\alpha x_3 + \sin\alpha x_2,\ \cos\alpha x_1 + \sin\alpha x_3,\ \cos\alpha x_2 + \sin\alpha x_1), \tag{5.1}$$

where A and α are any positive constants, the first of which is the amplitude of the field, and the second is its inverse spatial scale. The magnetic field $\boldsymbol{h}_0$ is a particular case of the spatially periodic field called a force-free Beltrami field [15, 16].

It is assumed that the quiescent medium considered is homogeneous, i.e.,

$$\rho = \rho_0 = \text{const}, \qquad p = p_0 = \text{const}, \qquad s = s_0 = \text{const}. \tag{5.2}$$

$$\rho = \rho_0 = \text{const}, \qquad p = p_0 = \text{const}, \qquad s = s_0 = \text{const}. \quad (5.2)$$

In addition, a certain point O is fixed in the boundless space as the origin of the Cartesian coordinate system. In this system, the domain shaped like a rectangular parallelepiped is selected geometrically:

$$\Omega = \left\{(x_1, x_2, x_3):\ -\frac{\pi}{2\alpha} \leqslant x_1 \leqslant \frac{\pi}{2\alpha};\ -a \leqslant x_2 \leqslant a;\ 0 \leqslant x_3 \leqslant \frac{\pi}{2\alpha}\right\}. \tag{5.3}$$

Here a is a certain positive constant.

It is possible to convince oneself that the quiescent state (5.1), (5.2) is the exact solution of the steady-state equations (1.6), and the field $\boldsymbol{h}_0$ satisfies condition (1.7) on the parallelepiped surface Ω.

This quiescent state will be unstable to, for example, the field of Lagrangian displacements, which has the following form at the initial moment:

$$\boldsymbol{\xi}^0 = b(x_2^2, 0, 0) \tag{5.4}$$

(b is a constant positive factor).

Indeed, the direct calculations with the use of (2.6) and (5.1)–(5.4) show that, in this case,

$$\Pi(0) = \frac{b^2 A^2 a^3}{2\pi\alpha^2}\Big[\frac{1}{3}(\pi^2+8) - \frac{1}{5}a^2\alpha^2\Big].$$

It follows that condition (2.7) is satisfied for

$$\alpha > \frac{1}{a}\Big[\frac{5}{3}(\pi^2+8)\Big]^{1/2}. \tag{5.5}$$

Choosing α and a in a way such that the inequality (5.5) holds, it is easy to see that the perturbations (5.4) imposed on the quiescent states (5.1) and (5.2) increase according to the two-sided exponential estimates (4.7), (4.12), and (4.14). Here, using formulas (4.10) and (4.13), one can determine the rate of increase of Λ^+ for the most rapidly growing perturbations in (5.4).

This example allow us to make some important remarks. First, condition (5.5) on the inverse spatial scale α of the field $\boldsymbol{h}_0$ (5.1) is in agreement with the conclusion of Molodensky [12–14] on the instability of a force-free Beltrami field to small large-scale perturbations. Secondly, the existence of the field of Lagrangian displacements $\boldsymbol{\xi}^0$ (5.4), relative to which the quiescent states (5.1) and (5.2) are unstable, indicates the incorrect result of [15] according to which the force-free Beltrami field should be absolutely steady against any small spatial perturbations. Makov [16] also indicated the incorrectness of this result; however, the example of the initial field of Lagrangian displacements for which the second variation of the magnetic-field energy with satisfaction of a definite condition becomes negative cannot be considered demonstrative, because this field contradicts the solenoidality requirement adopted by the author of [16].

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